DynStoch 2022

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29 June – 1 July

Book of abstracts

Index of speakers

S1. Wednesday, June 29, 9:30 - 10:30

1- INFERENCE FOR THE JANSEN-RIT MODEL

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Bayesian estimation; diffusion; EEG-data

The EEG is an important tool in clinical neurology and clinical neurophysiology. The Jansen-Rit model of cortical columns was proposed in [1] in 1995. It consists of 3 interacting second-order differential equations, say with components y_0, y_1 and y_2. The EEG-signal is modelled by $y_1 - y_2$. Recently, a stochastic version of this model was proposed in [2], leading to a system of stochastic differential equations.

Inference for this model is complicated as the likelihood is intractable, the SDE is hypo-elliptic and the drift is highly nonlinear. For this reason, Approximate Bayesian Computation was proposed in [2].. I will show how likelihood based inference can be done by exploiting the inherent structure of the Jansen-Rit model and using the results in [3]. This in turn yields an algorithm that targets not the ABC-posterior, but the actual posterior. The results also shed light on multimodality and practical identifiability if only discrete-time data are available.

References

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2- PROBABILISTIC PROPERTIES AND PARAMETRIC INFERENCE OF SMALL VARIANCE McKEAN-VLASOV STOCHASTIC DIFFERENTIAL EQUATIONS

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McKean-Vlasov stochastic differential equations; Continuous observations, Parametric inference; Small noise; Approximate likelihood

We consider a process $X(t)$ solution of a one-dimensional nonlinear self-stabilizing stochastic differential equation, with classical drift term $V(\alpha, x)$ depending on an unknown parameter α, self-stabilizing term $\Phi(\beta, x)$ depending on another unknown parameter β and small noise amplitude ε. Self-Stabilization is the effect of including a mean-field interaction in addition to the state-dependent drift. Adding this term leads to a nonlinear or Mac Kean-Vlasov Markov process with transitions depending on the distribution of X_t. We study the probabilistic properties of (X_t) as ε tends to 0 and exhibit a Gaussian approximating process for (X_t). Next, we study the estimation of (α, β) from a continuous observation of $(X_t, t \in [0, T])$. We build explicit estimators using an approximate log-likelihood function obtained from the exact log-likelihood function of a proxi-model. We prove that, for fixed T, as ε tends to 0, α can be consistently estimated with rate ε^{-1} but not β. Then, considering n i.i.d. sample paths $(X_t^i, i = 1, ..., n)$, we build consistent and asymptotically Gaussian estimators of (α, β) with rates $\sqrt{n}\varepsilon^{-1}$ for α and $\sqrt{n}$ for β. Finally, we prove that the statistical experiments generated by (X_t) and the proxi-model are asymptotically equivalent in the sense of the Le Cam Δ-distance both for the continuous observation of one path and for n i.i.d. paths under the condition $\sqrt{n}\varepsilon \to 0$, which justifies our statistical method.

Next, we study the estimation of the unknown parameters α, β from a continuous observation of $(X_t, t \in [0, T])$ under the double asymptotic framework ε tends to 0 and T tends to infinity. After centering and normalization of the process, uniform bounds with respect to $t \geq 0$ and ε are derived. We then build an explicit approximate log-likelihood leading to consistent and asymptotically Gaussian estimators under the condition $\varepsilon\sqrt{T} \to 0$ with original rates of convergence: the rate for the estimation of α is either ε^{-1} or $\sqrt{T}\varepsilon^{-1}$, the rate for the estimation of β is $\sqrt{T}$. Moreover, these estimators are asymptotically efficient.

Several examples illustrating the theory are finally proposed for the two asymptotics respectively $\varepsilon \to 0, T$ fixed and $\varepsilon \to 0, T \to \infty$.

References

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S2. Wednesday, June 29, 11:00 - 12:30

1- CHANGE-POINT DETECTION OF THE DRIFT PARAMETER OF A PARTICULAR CLASS OF 2-DIMENSION OU PROCESSES

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Change-point problem; Maximum likelihood estimator; Ornstein-Uhlenbeck process; Harmonic oscillator

In this work we investigate the problem of detecting if a parameter change actually occurs in the drift matrix of a particular class of 2-dimension Ornstein-Uhlenbeck processes with singular diffusion matrix. The framework considered includes the model for the dynamics of a billinear oscilator subject to stochastic loads [1]. A regime change of the OU process occurs when, at an unknown time, one of the parameters appearing in the drift matrix changes from a value $\theta_1 > 0$ to a value $\theta_2 > 0$, both values being unknown. Thus we consider a piecewise linear model that may switch at an unknown time. The stochastic forces acting on the system are assumed to be driven by a Wiener process. An hypothesis test that solves this problem has already been proposed for the billinear oscilator, in previous work [3]. The test statistic used is a likelihood ratio and the idea behind the test closely follows [2]. We give detailed proofs of the results established concerning the asymptotic distribution of the test statistic. We present a simulation study that illustrates the properties of the test and we analyze in particular the influence of the parameter values in the test performance.

References

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2- NADARAYA-WATSON ESTIMATOR FOR I.I.D. PATHS OF DIFFUSION PROCESSES

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Diffusion processes; Nonparametric drift estimation; Nadaraya-Watson estimator; PCO method; Cross validation.

Consider $N, d \in \mathbf{N}^*$, $x_0 \in \mathbf{R}$, $T > 0$ and N independent copies $X^1, \dots, X^N$ of the diffusion process X defined by

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \ ; \ t \in [0, T],$$

where b and σ are continuous functions from $\mathbf{R}$ into itself, and W is a Brownian motion.

The estimation of the drift function b from continuous-time and discrete-time observations of $X^1, \dots, X^N$ is a functional data analysis problem already investigated in the parametric framework (see Ditlevsen and De Gaetano (2005), Overgaard et al. (2005), Comte, Genon-Catalot and Samson (2013), etc.) and more recently in the nonparametric framework (see Comte and Genon-Catalot (2020), Della Maestra and Hoffmann (2021), Denis et al. (2021), etc.). Our talk will deal with the results of Marie and Rosier **[1]**.

Consider the continuous-time Nadaraya-Watson estimator

$$\widehat{b}_{N,h,\eta}(x) := \frac{\dfrac{1}{N(T-t_0)h}\displaystyle\sum_{i=1}^{N}\int_{t_0}^{T} K\left(\frac{X_t^i - x}{h}\right) dX_t^i}{\dfrac{1}{N(T-t_0)\eta}\displaystyle\sum_{i=1}^{N}\int_{t_0}^{T} K\left(\frac{X_t^i - x}{\eta}\right) dt} \ ; \ x \in \mathbf{R}$$

of b, where $t_0 \in (0, T]$ and $K : \mathbf{R} \to \mathbf{R}$ is a kernel.

During this talk, we will present a risk bound on $\widehat{b}_{N,h,\eta}(.)$, a risk bound on a discrete-time approximate of our Nadaraya-Watson estimator, an oracle inequality for an adaptive estimator obtained via a PCO type bandwidth selection method, and some numerical experiments for an adaptive estimator obtained via a leave-one-out cross validation type procedure.

References

[1] N. Marie and A. Rosier. *Nadaraya-Watson Estimator for I.I.D. Paths of Diffusion Processes.* In revision, 2021.

3- CLASSIFICATION PROCEDURE FOR DIFFUSION PATHS

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Supervised classification; diffusion processes; non-parametric estimation; plug-in classifier

Recent advents in modern technology have generated labeled data recorded at high frequency, that can be modelled as functional data. This work focuses on multiclass classification problem for functional data modelled by a stochastic differential equation. Few works study the case where functional data are modelled by diffusion processes, which is why the construction of classification procedures adapted to this type of model is a major challenge. We focus on time-homogeneous stochastic differential equations with unknown and non-parametric drift and diffusion coefficients. The objective is to propose an implementable *plug-in* classification procedure based on a non-parametric estimation from a learning sample of drift and diffusion coefficients by minimizing a least squares contrast on a **B**-spline basis. We then establish the consistency of the obtained empirical classifier.

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S3. Wednesday, June 29, 14:00 - 15:30

1- MULTIPLICATIVE DECONVOLUTION IN A BIVARIATE STOCHASTIC VOLATILITY MODEL

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Stochastic Volatility Model; Nonparametric Density Estimation; Multiplicative Deconvolution; Adaptivity

In this talk, we construct a nonparametric estimator of the density f_{σ^2} of a strictly stationary, bivariate volatility process $(\sigma_t^2)_{t\geq 0}$ based on discrete observations of the process $(Y_t)_{t\geq 0}$, the solution of the stochastic differential equation

$$dY_t = \Sigma_t dW_t, \quad \Sigma_t = \mathrm{diag}(\sigma_{t,1}, \sigma_{t,2}), \tag{1}$$

where $(W_t)_{t\geq 0}$ is a bivariate Brownian motion independent of $(\sigma_t)_{t\geq 0}$. More precisely, we consider a bivariate generalisation of the stoachstic volatility model considered in [1], identify the underlying multiplicative measurement error model and build an estimator using the results of [2].
Based on the estimation of the Mellin transform of the density f_{σ^2} and a spectral cut-off regularisation of the inverse Mellin transform, we propose a fully data-driven density estimator where the anisotropic choice of the spectral cut-off parameter is dealt by a model selection approach. Furthermore, we study the risk of our estimator and show its adaptivity up to a negligible term. We demonstrate the reasonable performance of our estimator using a Monte-Carlo simulation and consider several examples for the volatility processes $(\sigma_t^2)_{t\geq 0}$.

References

[1] Comte, F., Genon-Catalot, V. (2006) *Penalized Projection Estimator for Volatility Density*, Scandinavian journal of statistics, 33(4), 875-893.

[2] Brenner Miguel, S. (2021) *Anisotropic spectral cut-off estimation under multiplicative measurement errors*, arXiv preprint arXiv:2107.02120

2- ON THE RATE OF ESTIMATION FOR THE STATIONARY DISTRIBUTION OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH AND WITHOUT JUMPS

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Non-parametric estimation; Minimax convergence rate; Ergodic diffusion; Anisotropic density estimation

In this talk, we discuss some results on the estimation of the invariant density associated to a multivariate diffusion $X = (X_t)_{t\geq 0}$, solution of a stochastic differential equation with or without jumps.
The estimation of the invariant density is a problem of great relevance which has a huge amount of applications in physics and numerical methods (the Markov Chain Monte Carlo, among them). Evidence of the attractiveness of the non-parametric estimation for the stationary measure of a diffusion is the fact that such a subject is both a long-standing problem and a living topic.
We propose kernel density estimators based on the continuous record of the trajectory $X^T = (X_t)_{0\leq t\leq T}$, and we measure their accuracy by studying the size of their pointwise L^2 error.
We first of all find the convergence rates associated to the proposed estimators. After that, we wonder if it is possible to propose other estimators which achieve better convergence rates.

3- LOCALLY STATIONARY PROCESSES IN CONTINUOUS-TIME

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asymptotic normality; consistency; Lévy-driven state space models; locally stationary; M-estimation; stationary approximations; weak dependence

Discrete-time locally stationary stochastic processes constitute a class of approaches to model non-stationary time series and find application throughout various disciplines (see e.g. [1] for an overview). Despite their success, these approaches have mostly been carried out for models defined on $\mathbb{Z}$, i.e. in a discrete-time framework. Surprisingly, there is so far no general theory for locally stationary models defined on $\mathbb{R}$, i.e. in a continuous-time framework, available.
In this talk, we tackle this issue and introduce a theory on stationary approximations for locally stationary continuous-time processes. Based on the stationary approximation, we use θ-weak dependence to establish laws of large numbers and central limit type results under different observation schemes (see [2]). Hereditary properties for a large class of finite and infinite memory transformations show the flexibility of the developed theory. In particular, we outline possible applications of our theory to model non-stationary extremes.
In addition, we provide inference methodologies suitable for our non-stationary setting by establishing asymptotic properties of M-estimators (see [3]). As examples we consider different time-varying continuous-time models including time-varying Ornstein-Uhlenbeck processes, for which we introduce a localized Least-Squares estimator. For observations from a time-varying Lévy-driven state space model we establish a localized quasi maximum likelihood estimator. A simulation study shows the applicability of the estimation procedures.

References

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S4. Wednesday, June 29, 16:00 - 18:00

1- EFFICIENT INFERENCE FOR HIGH-FREQUENCY DATA

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Local Asymptotic Normality; Maximum Likelihood Estimator; fractional Gaussian noise; One-step estimation procedure

The theory of Local Asymptotic Normality (LAN) provides a powerful framework under which the asymptotic optimality of estimators and tests of hypothesis for finite-dimensional parameters can be studied. When the LAN property holds true for a statistical experiment with a non-singular Fisher information matrix, minimax theorems can be applied and a lower bound for the variance of the estimators can be derived. Moreover, the asymptotic power of a test of hypothesis can be evaluated by a computation under the null hypothesis. Although a lot of attention has been paid to the high-frequency data due to their increasing availability in different applications, several statistical experiments under the high-frequency scheme remain not fully understood. For instance, at high frequency, the variance and the Hurst exponent for the fractional Gaussian noise (fGn) or the scale and the stability index for the stable Lévy process are melting. Weak LAN property with a singular Fisher information matrix was obtained in [1] for the fGn and in [2] for the stable Lévy process. Due to this singularity, no minimax theorem can be applied and it has been unclear for long time whether the maximum likelihood estimator (MLE) possess any kind of asymptotic optimality property. Then, a non-singular LAN property has been established for fGn in [3] and for the stable Lévy process in [4] allowing to define the asymptotic optimality in these statistical experiments. The LAN theory does not directly provide the construction of efficient estimators. In regular statistical experiments, the MLE generally achieves optimality. Nevertheless, its computation is time consuming, and alternatives should be found to handle big or high-frequency dataset available in the fields of insurance and finance. In this direction, the Le Cam one-step estimation can be proposed. In this presentation, the asymptotically efficient and fast estimation procedures used in [5] for the fGn and in [6] for the stable Lévy process will be recalled and recent results, that will be discussed within the ANR project EFFI, will be introduced.

References

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2- PARAMETER ESTIMATION FOR HIDDEN MARKOV PROCESSES

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Estimation; Kalman filter; small noise; large samples

We present a survey of several results related with the partially observed linear systems like

$$\begin{aligned} \mathrm{d}X_t &= f_t(\vartheta)\, Y_t \mathrm{d}t + \varepsilon_1 \sigma_t \mathrm{d}W_t, \quad X_0 = 0, \quad 0 \le t \le T, \\ \mathrm{d}Y_t &= a_t(\vartheta)\, Y_t \mathrm{d}t + \varepsilon_2 b_t(\vartheta)\, \mathrm{d}V_t, \quad Y_0 = y_0 \end{aligned}$$

Here W_t and V_t are independent Wiener processes, $f_t(\cdot), a_t(\cdot), b_t(\cdot)$ and σ_t are known functions and the unknown parameter $\vartheta \in \Theta = (\alpha, \beta)$.

We describe the properties of the MLE, BE, One-step MLE and One-step MLE-processes in different asymptotics, like a)$\varepsilon_1 = \varepsilon_2 = \varepsilon \to 0$, b)$\varepsilon_1 \to 0, \varepsilon_2 = 1$, c)$\varepsilon_1 = \varepsilon_2 = 1, T \to \infty$.

References

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[2] Kutoyants, Yu. A. (2020) *Hidden Markov model where higher noise makes smaller errors*, submitted

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3- LINEAR FILTERING WITH FRACTIONAL NOISES: LARGE TIME AND SMALL NOISE ASYMPTOTICS

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stochastic filtering; fractional Brownian motion; asymptotic analysis

Classical state-space approach to optimal estimation of stochastic processes is efficient when the driving noises are generated by martingales. In particular, the weight function of the optimal linear filter, which solves a complicated operator equation in general, simplifies to the Riccati ordinary differential equation in the martingale case. This reduction lies in the foundations of the Kalman-Bucy approach to linear optimal filtering. In this talk we consider a basic Kalman-Bucy model with noises, generated by independent fractional Brownian motions, and develop a new method of asymptotic analysis of the integro-differential filtering equation arising in this case. We establish existence of the steady-state error limit and find its asymptotic scaling in the high signal-to-noise regime. Closed form expressions are derived in a number of important cases.

References

[1] D. Afterman, P. Chigansky, M. Kleptsyna, D. Marushkevych (2022) *Linear filtering with fractional noises: large time and small noise asymptotics*, to appear in SIAM Journal on Control and Optimization (preprint arXiv:1911.10062)

4- ESTIMATION OF THE HURST PARAMETER FROM CONTINUOUS NOISY DATA

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fractional Brownian motion; parameter estimation; Local Asymptotic Normality (LAN)

The talk presents a contribution to the problem of asymptotic minimax estimation of the Hurst parameter of the fractional Brownian motion (fBm). Previously, this problem has been studied when the fBm trajectory is observed on a discrete grid of points, either directly [1] or partially, with additive noise [4]. We consider the the setting in which it is observed continuously in time with noise modeled by an independent Brownian motion. When the Hurst parameter is greater than 3/4 consistent estimation from discrete data is no longer possible, [2] [5]. The two relevant asymptotic regimes are either when the length of the observation interval increases to infinity or intensity of the noise decreases to zero. The main result is a proof of local asymptotic normality (LAN) in both of these asymptotic regimes, which reveals the optimal minimax rates.

References

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S5. Thursday, June 30, 9:00 - 10:30

1- SIMPLIFIED QUASI-LIKELIHOOD ANALYSIS

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Ibragimov-Khasminskii theory; Quasi-likelihood analysis; Polynomial type large deviation

The asymptotic decision theory by Le Cam and Hájek has been given a lucid perspective by the Ibragimov-Khasminskii theory on convergence of the likelihood random field. Their scheme has been applied to stochastic processes by Kutoyants. This program ensures that asymptotic properties of an estimator follow directly from the convergence of the random field if a large deviation estimate exists. The quasi-likelihood analysis (QLA) presented a polynomial type large deviation (PLD) inequality to go through a bottleneck of the program for analysis of nonlinear stochastic processes under various sampling schemes. A conclusion of the QLA is that if the quasi-likelihood random field is asymptotically quadratic and if a key index reflecting identifiability the random field has is non-degenerate, then the PLD inequality is always valid, and as a result, the program runs. Many studies already took advantage of the QLA theory. However, not a few of them are using it in an inefficient way yet. The aim of this paper is to provide a reformed and simplified version of the QLA and to improve accessibility to the theory. As an example of the effects of the program and the PLD, the user can obtain asymptotic properties of the quasi-Bayesian estimator by only verifying non-degeneracy of the key index. Yoshida (2022) lists various applications of the QLA.

References

[1] Yoshida, N. (2011) *Polynomial type large deviation inequalities and quasi-likelihood analysis for stochastic differential equations*, Annals of the Institute of Statistical Mathematics 63, 431–479.

[2] Yoshida, N. (2021) *Simplified quasi-likelihood analysis for a locally asymptotically quadratic random field*, arXiv:2102.12460.

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2- FORMULAE FOR COMPARING ERGODIC SDE MODELS

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Ergodic Lévy driven SDE; High-frequency sampling; Information criteria; Quasi-likelihoods

We consider relative model comparison for the parametric coefficients of an ergodic Lévy driven model observed at high-frequency for a long term, with adopting the fully explicit two-stage Gaussian quasi-likelihood function (GQLF) of the Euler-approximation type. The two types of information criteria: Gaussian quasi-AIC (GQAIC) and Gaussian quasi-BIC (GQBIC) are proposed through the stepwise procedure, and their theoretical properties are studied. In particular, we will show the following.

- The *mixed-rates structure* of the original GQLF affects the selection of the scale coefficient in an essential way, making the information-criteria formulae non-standard and quantitatively clarifying the relation between estimation precision and the sampling frequency for both GQAIC and GQBIC;
- It suggests that not joint (for drift and scale) but *stepwise* selection should be executed; this point turns out to be essential to compose explicit information criteria;
- For the GQBIC (Schwarz's BIC-type statistics) to have the model-selection consistency, one needs to properly *heat up* the Gaussian quasi-likelihood for the scale coefficient.

Numerical experiments together with implementation in the YUIMA R package [YUIMA14] are given to illustrate our theoretical findings.

We will also discuss related problems such as the potential use of the non-Gaussian quasi-likelihood under ergodicity.

Not a few parts of this talk are based on the joint work [EM22] with Shoichi Eguchi (Osaka Institute of Technology).

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3- LOCAL ASYMPTOTIC PROPERTIES FOR COX-INGERSOLL-ROSS PROCESS WITH DISCRETE OBSERVATIONS

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Cox-Ingersoll-Ross process; local asymptotic (mixed) normality property; local asymptotic quadraticity property; Malliavin calculus; parametric estimation; square root coefficient.

In this work, we consider a one-dimensional Cox-Ingersoll-Ross (CIR) process whose drift coefficient depends on unknown parameters. Considering the process discretely observed at high frequency, we prove the local asymptotic normality property in the sub- critical case, the local asymptotic quadraticity in the critical case, and the local asymptotic mixed normality property in the supercritical case. To obtain these results, we use the Malliavin calculus techniques developed recently for CIR process together with the Lp-norm estimation for positive and negative moments of the CIR process. In this study, we require the same conditions of high frequency $\Delta_n \to 0$ and infinite horizon $n\Delta_n \to \infty$ as in the case of ergodic diffusions with globally Lipschitz coefficients. However, in the non-ergodic cases, additional assumptions on the decreasing rate of Δ_n are required due to the fact that the square root diffusion coefficient of the CIR process is not regular enough.

S6. Thursday, June 30, 11:00 - 12:30

1- MIXING CONVERGENCE OF LSE FOR SUPERCRITICAL GAUSSIAN AR(2) PROCESSES USING RANDOM SCALING

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Autoregressive process; Least squares estimator; Mixing convergence; Stable limit theorem

For AR(1) processes, Häusler and Luschgy [3, Chapter 9] proved stable and mixing convergence of the least squares estimator (LSE) for the autoregressive parameter. Since stable (mixing) convergence yields convergence in distribution, these results immediately imply convergence in distribution of the LSE in question.

According to our knowledge, results on stable (mixing) convergence of the LSE for the autoregressive parameters of higher-order AR processes are not available in the literature. This motivates our study. We establish mixing convergence of the LSE for the autoregressive parameters of supercritical Gaussian AR(2) processes having real characteristic roots with different absolute values. We use an appropriate random scaling such that the limit distribution is a two-dimensional normal distribution concentrated on a one-dimensional ray determined by the characteristic root having the larger absolute value.

The proof is based on a multidimensional analogue of a one-dimensional stable limit theorem due to Häusler and Luschgy [3, Chapter 7] for so called explosive stochastic processes. In fact, our multidimensional analogue may be interesting in its own right as well. If time permits, then, as special cases, we present multidimensional stable limit theorems involving multidimensional normal-, Cauchy- and stable distributions as well.

The talk is based on the papers Barczy and Pap [1] and Barczy, K. Nedényi and Pap [2].

References

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2- OPTIMAL ESTIMATION OF THE HURST PARAMETER OF A ROUGH DIFFUSION OBSERVED AT DISCRETE TIME WITH AN ADDITIVE NOISE

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Scaling exponent; High frequency data; Fractional Brownian motion; Whittle estimator; LAN Property.

In this talk, we study the estimation of the the Hurst parameter in rough diffusion models, i.e. when the Hurst parameter $H \in (0, 1)$ can be close to 0. Such models are becoming more and more important in statistical finance since the seminal results of Gatheral, Jaisson and Rosenbaum [2].

The underlying Fractional Brownian motion is observed at discrete time with an additive or multiplicative noise. We will present a complete picture in the case of an additive noise, in the same spirit as [1] and will will address some preliminary results in the multiplicative case if time permits. Based on observing $\sigma W^H_{i/n} + \tau_n \varepsilon^n_i$, with $\tau_n \to 0$ or $\tau_n \approx 1$, we construct a Whittle estimator satisfying a central limit theorem with convergence rate depending on the noise level. It is $\sqrt{n}$ when $\tau_n n^H \ll 1$ and $(n/\tau_n^2)^{1/(4H+2)}$ otherwise. We also prove that this rate is minimax in both cases, by establishing the LAN property when $\tau_n n^H \ll 1$ and an adequate construction of the fractional Brownian motion in the second case.

References

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[2] Gatheral, J., Jaisson, T., Rosenbaum, M. (2018) *Volatility Is Rough* , Quantitative Finance, Vol. 18, No. 6, 933-949

3- ESTIMATING MIXED FRACTIONAL STABLE PROCESSES FROM HIGH-FREQUENCY DATA

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high frequency data; linear fractional stable motion; Lévy processes; selfsimilar processes

The linear fractional stable motion generalizes two prominent classes of stochastic processes, namely stable Lévy processes, and fractional Brownian motion. For this reason it may be regarded as a basic building block for continuous time models. We study a stylized model consisting of a superposition of independent linear fractional stable motions and our focus is on parameter estimation of the model. Applying an estimating equations approach, we construct estimators for the whole set of parameters and derive their asymptotic normality in a high-frequency regime. The conditions for consistency turn out to be sharp for two prominent special cases: (i) for Lévy processes, i.e. for the estimation of the successive Blumenthal–Getoor indices [1], and (ii) for the mixed fractional Brownian motion introduced by Cheridito [2]. In the remaining cases, our results reveal an interesting interplay between the Hurst parameter and the index of stability.

References

[1] Ait-Sahalia, Y., Jacod, J. (2012) *Identifying the successive Blumenthal-Getoor indices of a discretely observed process* Ann. Stat. 40(3):1430–1464.

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S7. Thursday, June 30, 14:00 - 15:30

1- ERGODICITY AND PROPAGATION OF CHAOS FOR MCKEAN-VLASOV SDES WITH LÉVY NOISE

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McKean-Vlasov SDEs; ergodicity; propagation of chaos; coupling

In this talk, based on the paper [3], we are concerned with stochasic differential equations (SDEs) of McKean-Vlasov type of the form

$$\begin{cases} dX_t = b(X_t, \mu_t)\,dt + dZ_t, X_0 \sim \mu_0, \\ \mu_t = \mathrm{Law}(X_t), \end{cases} \tag{2}$$

where $b : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ is a measurable function and $(Z_t)_{t\geq 0}$ is a d-dimensional Lévy process. Here $\mathcal{P}(\mathbb{R}^d)$ denotes the family of all probability measures on $\mathbb{R}^d$. We will present a construction of a coupling of solutions to such SDEs, and then we will apply it to study their convergence rates to stationary distributions. We will also consider a propagation of chaos result in the case where the drift b in (2) is given by

$$b(x, \mu) = b_1(x) + \int b_2(x, z)\,\mu(dz)$$

for some measurable functions $b_1 : \mathbb{R}^d \to \mathbb{R}^d$ and $b_2 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, in which case the solution to (2) can be approximated by an interacting particle system

$$dX_t^{i,n} = b_1(X_t^{i,n})\,dt + \frac{1}{n}\sum_{j=1}^{n} b_2(X_t^{i,n}, X_t^{j,n})\,dt + dZ_t^i, \quad i = 1, \ldots, n$$

driven by independent Lévy processes $(Z_t^i)_{t\geq 0}$. By combining our results with those from [1,2], we can also cover the case of SDEs of the form $dX_t = b(X_t, \mu_t)\,dt + \sigma(X_t)dW_t + dZ_t$ or $dX_t = b(X_t, \mu_t)\,dt + dW_t + \sigma(X_t)dZ_t$, where $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $(W_t)_{t\geq 0}$ is the standard Brownian motion in $\mathbb{R}^d$ and $(Z_t)_{t\geq 0}$ is a pure jump Lévy process independent of $(W_t)_{t\geq 0}$.

The presented approach, unlike more traditional methods, allows us to obtain explicit sharp bounds on convergence rates, measured in Wasserstein distances, which is relevant for applications in statistics and numerical analysis.

References

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2- A PROJECTION ESTIMATOR FOR NONPARAMETRIC DRIFT INFERENCE OF MCKEAN-VLASOV EQUATIONS (canceled)

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Nonparametric Estimation; Projection Estimators; McKean-Vlasov SDEs

We consider real-valued solutions of the McKean-Vlasov stochastic differential equation and construct a projection least-squares estimator for the underlying drift function in the regime where the amount of particles tends to infinity and the observed time horizon is fixed. We derive estimation rates both in the natural norm arising from the problem as well as the usual L^2 norm. The estimation procedure can also be applied to the associated particle system.

References

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[2] Comte, F., Genon-Catalot, V. (2020) *Nonparametric drift estimation for i.i.d. paths of stochastic differential equations*, The Annals of Statistics 48, 3336-3365.

3- SEMIPARAMETRIC ESTIMATION OF McKEAN-VLASOV SDEs

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Deconvolution; McKean-Vlasov SDEs; Mean field models; Minimax bounds; Semiparametric estimation

In this talk we study the problem of semiparametric estimation for a class of McKean-Vlasov stochastic differential equations. Our aim is to estimate the drift coefficient of a MV-SDE based on observations of the corresponding particle system. We propose a semiparametric estimation procedure and derive the rates of convergence for the resulting estimator. We further prove that the obtained rates are essentially optimal in the minimax sense.

References

[1] Belomestny, D., Pilipauskaitė, V., Podolskij, M. (2022+) *Semiparametric estimation of McKean-Vlasov SDEs*, Accepted for publication in Ann. Inst. H. Poincaré Probab. Statist.

S8. Thursday, June 30, 16:00 - 18:00

1- PARAMETRIC DRIFT ESTIMATION FOR HIGH DIMENSIONAL DIFFUSIONS

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Gabriela Ciolek *Paris, France*

drift; Lasso; high dimensional statistics; parameter estimation

In this talk we present a new parametric estimation method for the drift of a high dimensional diffusion model. More specifically, the dimension of the parameter space and the model are potentially large, and we develop the theory for a Lasso type estimator. The main results are based upon concentration inequalities for diffusion processes and elements of empirical processes theory.

2- SPARSE MARKOV MODELS FOR HIGH-DIMENSIONAL INFERENCE

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Daniel Y. Takahashi *Federal University of Rio Grande do Norte, Brazil*

Markov Chains; High-dimensional inference; Mixture Transition Distribution

Consider a sample of size n of a finite order Markov chain. In this full generality, we can only estimate the parameters of the Markov chain (the order d and the transition probabilities) in the regime $d = \mathcal{O}(\log(n))$, limiting the practical application of theses chains to small orders only. In this talk, we will discuss a way to overcome this constraint in a large class of Markov chains, namely the Mixture of Transition Distribution (MTD) models. In our main result, we will show that it is possible to select *a priori* the portion of the past that is relevant for the transition probabilities of a MTD, allowing the estimation of the model parameters even in the regime $d = \mathcal{O}(n)$. The practical performance of our estimation procedure will be illustrated through simulations.

References

[1] Ost, G. and Takahashi, D.Y. (2022) *Sparse Markov Models for High-Dimensional Inference*, Arxiv.

3- CENTRAL LIMIT THEOREMS FOR STATIONARY RANDOM FIELDS UNDER WEAK DEPENDENCE WITH APPLICATION TO AMBIT AND MIXED MOVING AVERAGE FIELDS

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Bennet Ströh *Imperial College, United Kingdom*

Stationary random fields; weak dependence; central limit theorems; mixed moving average fields; CARMA fields; ambit fields

We obtain central limit theorems for stationary random fields employing a novel measure of dependence called θ-lex weak dependence. We show that this dependence notion is more general than strong mixing, i.e., it applies to a broader class of models. Moreover, we discuss hereditary properties for θ-lex and η-weak dependence and illustrate the possible applications of the weak dependence notions to the study of the asymptotic properties of stationary random fields. Our general results apply to mixed moving average fields (MMAF in short) and ambit fields. We show general conditions such that MMAF and ambit fields, with the volatility field being an MMAF or a p-dependent random field, are weakly dependent. For all the models mentioned above, we give a complete characterization of their weak dependence coefficients and sufficient conditions to obtain the asymptotic normality of their sample moments. Finally, we give explicit computations of the weak dependence coefficients of MSTOU processes and analyze under which conditions the developed asymptotic theory applies to CARMA fields.

4- SIMULATION METHODS FOR TRAWLS PROCESSES AND AMBIT FIELDS

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Stochastic Simulation; Numerical Study of Stochastic Processes

Trawl processes are continuous-time, stationary and infinitely divisible processes which can describe a wide range of possible serial correlation patterns in data. This talk introduces a new algorithm for the efficient and exact simulation of monotonic trawl processes. The algorithm accommodates any monotonic trawl shape and any infinitely divisible distribution described via the Lévy seed, requiring only access to samples from the distribution of the Lévy seed. We describe how the above method can be generalized to a simulation scheme for monotonic ambit fields via Monte Carlo methods.

References

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[2] Barndorff-Nielsen, O. E., Benth, F. E., Veraart, A. E. (2018) Ambit Stochastics, Springer.

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S9. Friday, July 1st, 9:00 - 10:30

1- PARAMETER ESTIMATION FOR LINEAR PARABOLIC SPDES IN TWO SPACE DIMENSIONS FROM DISCRETE OBSERVATIONS

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Kaino, Yusuke *Kobe University, Japan*

Adaptive estimation; high frequency data; stochastic partial differential equations in two space dimensions; Q-Wiener process

We consider parameter estimation for a linear parabolic second-order stochastic partial differential equation (SPDE) in two space dimensions driven by a Q-Wiener process based on high frequency data in time and space. There are several studies on parametric estimation of a linear parabolic second-order SPDE in one space dimension driven by the cylindrical Wiener process based on high frequency data observed on a fixed region $[0, T] \times [0, 1]$ with T being the last observation time, see [1] and [2]. They derived minimum contrast estimators for unknown parameters of the SPDE in one space dimension and proved the asymptotic normality of the estimators. In this talk, we first obtain minimum contrast estimators for unknown parameters of the coordinate process of the linear parabolic second-order SPDE in two space dimensions based on the thinned data with respect to space. Then, by using the minimum contrast estimators, an approximate coordinate process of the SPDE in two space dimensions is constructed. Finally, utilizing the approximate coordinate process based on the thinned data with respect to time, we derive adaptive estimators of the coefficient parameters of the SPDE in two space dimensions. It is shown that the adaptive estimators have asymptotic normality under some regularity conditions. Numerical simulations of the proposed estimators are also given.

References

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[2] Hildebrandt, F., Trabs, M. (2021). *Parameter estimation for SPDEs based on discrete observations in time and space*, Electronic Journal of Statistics, 15, 2716–2776.

2- CALIBRATION OF SPDES BASED ON DISCRETE OBSERVATIONS: SEMILINEAR AND MULTIDIMENSIONAL EQUATIONS

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Trabs, Mathias *Karlsruhe Institute of Technology, Germany*

stochastic partial differential equation; infill asymptotics; realized quadratic variation; nonparametric estimation

In view of a growing number of stochastic partial differential equation (SPDE) models used in the natural sciences as well as in mathematical finance, their data-based calibration has become an increasingly active field of research during the last few years. While the analysis of functional observations of SPDEs has proven to be manageable for quite general equations, the more realistic scenario of discrete observations in time and space was mainly limited to linear, one-dimensional equations, for which explicit computations are possible. In the talk, we present ways of overcoming these limitations when high-frequency observations in time and space are available.

In the first part of the talk, we consider one-dimensional reaction-diffusion equations. Exploiting the higher order regularity of the nonlinear component of the solution, we show that the asymptotic properties of diffusivity and volatility estimators derived from realized quadratic variations in the linear setup generalize to the semilinear SPDE. In particular, we obtain a rate-optimal joint estimator of the two parameters. Also, we derive a nonparametric estimator for the reaction function specifying the underlying equation. The estimate is chosen from a finite-dimensional function space based on a least squares criterion. Oracle inequalities with respect to the L^2-risk provide conditions for the estimator to achieve the usual nonparametric convergence rate.

The second part of the talk addresses parameter estimation for multidimensional linear systems. Inspired by the local obervations approach by Altmeyer & Reiß [3], we employ tools from semigroup theory to derive and analyze method of moments estimators for diffusivity and volatility parameters. Doing so, our estimators do not depend on the geometry of the spatial domain and, in particular, we avoid working with the spectral decomposition of the solution process. In line with the one-dimensional case, we end up in different regimes depending on the interplay of spatial and temporal sampling frequencies with a special role being played by the parabolic sampling design, where (temporal frequency)=(spatial frequency)^2.

References

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[2] Hildebrandt, F., Trabs, M. (2021) *Parameter estimation for SPDEs based on discrete observations in time and space*, Electron. J. Stat. 15(1), 2716-2776.

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3- INFILL ASYMPTOTICS FOR FUNCTIONAL OBSERVATIONS OF SPDEs

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Power Variations; SPDE; Functional Data Analysis; Volatility

For spatio-temporal stochastic processes, volatility can be interpretet as a process of square-roots of instantaneous covariance operators and, thus, takes a pivotal role for describing smooth features underlying the dynamics. We present a feasible asymptotic distribution theory for the estimation of the integrated volatility operator $\int_0^t \Sigma_s ds := \int_0^t \sigma_s \sigma_s^* ds$ corresponding to a stochastic partial differential equation (SPDE) in a separable Hilbert space H of the form

$$dY_t = (\mathcal{A}Y_t + \alpha_t)dt + \sigma_t dW_t,$$

based on observations $Y_{i\Delta_n}, i = 1, ..., \lfloor t/\Delta_n \rfloor$, $\Delta_n = \frac{1}{n}$, of its mild solution (*functional data*). Here $\mathcal{A}$ is the generator of a strongly continuous semigroup $\mathcal{S} := (\mathcal{S}(t))_{t\geq 0}$ on H, W is a cylindrical Wiener process, α and σ are the drift- and Hilbert-Schmidt-valued volatility process respectively. Such SPDEs constitute a well-established framework for describing spatio-temporal dynamics with applications in, e.g., finance or physics (c.f. [1]). Our theory is based on the *semigroup adjusted realised covariation* ($SARCV$), given by

$$SARCV_t^n := \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \tilde{\Delta}_i^n Y^{\otimes 2} := \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(Y_{i\Delta_n} - \mathcal{S}(\Delta) Y_{(i-1)\Delta_n} \right)^{\otimes 2},$$

which was shown to be a consistent estimator of the integrated volatility $\int_0^t \Sigma_s ds$ in [2]. $h^{\otimes 2} = \langle h, \cdot \rangle h$ denotes the usual tensor product. Here we present also the follow-up work, which treats the derivation of a functional central limit theorem, showing that $\sqrt{n}(SARCV_t^n - \int_0^t \Sigma_s ds)$ converges stably in law as a process to a Gaussian process with marginal laws $\mathcal{N}(0, \Gamma_t)$. The covariance operator Γ_t, called the *asymptotic variance*, is unknown a priori, which is why we also provide a feasible estimator for the asymptotic variance by establishing laws of large numbers for more general *semigroup adjusted realised multipower variations* ($SAMPV$). In that way, we derive feasible central limit theorems.

Our approach can be understood as a generalisation of the asymptotic theory for power variations of finite-dimensional semimartingales (as presented for instance in [3]) to the Hilbert space framework. This is because the semigroup adjustment in the $SARCV$ just becomes relevant if $(Y_t)_{t\in[0,T]}$ is not an H-valued semimartingale, which is a purely infinite-dimensional issue. Such results may be desirable in many situations, such as for the evaluation of forecasting distributions or yield curve evolutions.

References

[1] Da Prato, G., Zabczyk, J. (2014) *Stochastic Equations in Infinite Dimensions*, Cambridge University Press.

[2] Benth, F.E., Schroers, D., Veraart, A.E.D. (2022) *A weak law of large numbers for realised covariation in a Hilbert space setting*, Stoch. Proc. Applic. 145, 241–268.

[3] Jacod, J., Protter, P. (2012) *Discretization of Processes*, Springer.

S10. Friday, July 1st, 11:00 - 12:30

1- PARAMETER ESTIMATION FOR SPDES WITH MULTIPLICATIVE NOISE

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Stochastic partial differential equation; Local measurements; Drift estimation; Stable convergence

We propose the stochastic heat equation driven by the multiplicative space-time white noise and we introduce several estimators to the diffusivity parameter that are inspired by the augmented maximum likelihood estimator from [3]. Since the observation scheme relies on local measurements, the asymptotics of the estimators is pursued with a fixed time horizon and with the spatial resolution of the observations tending to zero.

We show asymptotic normality of the proposed estimators using stable convergence results from [4]. The performance of the estimators is demonstrated in numerical and real data experiments.

References

[1] Altmeyer R., Bretschneider T., Janák J., Reiß M. (2022) *Parameter estimation in an SPDE model for cell repolarization*, Journal on Uncertainty Quantification, to appear, arXiv:2010.06340.

[2] Altmeyer R., Cialenco I., Pasemann G. (2020) *Parameter estimation for semilinear SPDEs from local measurements*, ArXiv preprint, arXiv:2004.14728.

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2- ESTIMATION FOR THE REACTION TERM IN SEMI-LINEAR SPDES UNDER SMALL DIFFUSIVITY

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Semi-linear SPDEs; Poincaré inequality; Maximum Likelihood Estimation

We propose a novel asymptotic regime for statistical inference for SPDEs, in which the diffusivity of the system tends to zero. The maximum likelihood estimator for the reaction term in semi-linear SPDEs is consistent, satisfies a central limit theorem and is asymptotically efficient. In contrast to most of the existing literature, we do not require that the time horizon increases to achieve consistency. A higher order of the leading differential operator results in a faster convergence rate. The fluctuations of the empirical Fisher information around its mean are controlled using the infinite-dimensional Poincaré inequality. Both parametric and non-parametric estimation is discussed. Numerical examples illustrate the key findings.

References

[1] Gaudlitz, S., Reiß, M. (Working paper) *Estimation for the reaction term in semi-linear SPDEs under small diffusivity*

3- PARAMETER ESTIMATION FOR SEMILINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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Diffusivity estimation; reaction-diffusion equations; maximum likelihood

The problem of parametric drift estimation for semilinear stochastic partial differential equations (SPDE) is considered based on a maximum–likelihood approach. The diffusivity of such models is estimated in finite time based on a single trajectory with high resolution in space. This is implemented by observing either a large number of Fourier modes (spectral approach), a large number of spatial point evaluations of the process (discretized spectral approach) or a convolution with a kernel of small diameter (local approach). Asymptotic properties of different estimators within these observation schemes are discussed, based on a spatial regularity analysis of the solution to the underlying SPDE. Examples of the general theory include reaction-diffusion equations, Burgers equation and equations of Cahn–Hilliard type. Special emphasis is put on the issue of model misspecification, with respect to either the drift or the driving noise.

The theoretical results are supported by a numerical simulation.

As an extension, the case of simultaneous diffusivity and reaction parameter estimation from spectral observations is treated in the context of stochastic activator-inhibitor models. This is applied to experimental observations of the actin marker concentration within *Dictyostelium discoideum* giant cells, whose spatiotemporal dynamics is described as a stochastic FitzHugh–Nagumo system. The performance of different estimators is compared on synthetic data from numerical simulation as well as real data.

The material presented in this talk is covered in the author's PhD thesis [1], which is partially based on the papers [2, 3, 4].

References

[1] Pasemann, G. (2021) *Parameter Estimation for Semilinear Stochastic Partial Differential Equations*, PhD thesis, Technical University Berlin.

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S11. Friday, July 1st, 14:00 - 15:30

1- ON SMOOTH CHANGE-POINT LOCATION ESTIMATION FOR POISSON PROCESSES AND SKOROKHOD TOPOLOGIES

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Inhomogeneous Poisson process; Smooth change-point; Maximum likelihood estimator; Bayesian estimators; Asymptotic efficiency; Ibragimov-Khasminskii method; Skorokhod topologies

We consider the problem of estimation of the location of what we call *smooth change-point* from n independent observations of an inhomogeneous Poisson process. The *smooth change-point* is a transition of the intensity function of the process from one level to another which happens smoothly, but over such a small interval, that its length δ_n is considered to be decreasing to 0 as $n \to +\infty$.

We study the maximum likelihood estimator (MLE) and the Bayesian estimators (BEs), and show that there is a “phase transition” in the asymptotic behavior of the estimators depending on the rate at which δ_n goes to 0. More precisely, if δ_n goes to zero slower than the “critical” rate $1/n$, the behavior resembles that of the smooth case (studied by Kutoyants in [4]), and if δ_n goes to zero faster than $1/n$, the behavior is exactly the same as in the change-point case (also studied by Kutoyants in [4]). We call these two situations *slow case* and *fast case*, respectively.

More specifically, we show that in the *slow case* our model is locally asymptotically normal (with a rather unusual rate $\sqrt{\delta_n/n}$), and the MLE and the BEs are consistent, asymptotically normal and asymptotically efficient. As to the *fast case*, the MLE and the BEs are consistent, converge at the rate $1/n$, have different limit distributions, and only the BEs are asymptotically efficient.

It should be noted that all these results were obtained using the likelihood ratio analysis method developed by Ibragimov and Khasminskii in [3], which equally yields the convergence of polynomial moments of the considered estimators. On the other hand, for the study of the MLE, this method needs the convergence of the normalized likelihood ratio in some functional space, and up to the best of our knowledge, until now it was only applied using either the space of continuous functions equipped with the topology induced by the sup norm, or the space of càdlàg functions equipped with the usual Skorokhod topology (called J_1 by Skorokhod himself). However, we will see that in the *fast case* this convergence can not take place in neither of these topologies. So, the results concerning the MLE in the *fast case* were obtained by first extending the Ibragimov-Khasminskii method to use a weaker topology M_1 (also introduced by Skorokhod in his seminal paper [5], along with J_1 and two other topologies).

Finally, let us note that these results will be included in the future Ph.D thesis [1], and that the results concerning the *slow case* and the BEs in the *fast case* were already published in [2].

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2- ADAPTIVE ESTIMATION OF THE JUMP COEFFICIENT OF A JUMP DIFFUSION

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Adaptive estimation, jump diffusion

We consider a jump diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \xi(X_{t^-})dL_t, \quad X_0 = \eta$$

with $(W_t)_{t\geq 0}$ a Brownian motion, $(L_t)_{t\geq 0}$ a pure jump Lévy process of Lévy measure ν independant of $(W_t)_{t\geq 0}$ and η a random variable independant of $(W_t)_{t\geq 0}$ and $(L_t)_{t\geq 0}$. This process is observed at discrete times $t = 0, \Delta, \ldots, n\Delta$ with Δ small and $n\Delta$ large. We assume that the process $(X_t)_{t\geq 0}$ is stationnary, ergodic and exponentially β-mixing. Our aim is to estimate the jump coefficient $\xi(x)$ on the compact set A. A solution is to estimate the function $g(x) = \sigma^2(x) + \xi^2(x)$, and the diffusion coefficient $\sigma^2(x)$, to deduce the jump coefficient $\xi^2(x)$ by substraction. But the convergence of this estimator depends on the regularity of both σ^2 and ξ^2, and is more difficult if the Lévy measure is not finite. Therefore, we consider the increments

$$T_{k\Delta} = \frac{(X_{(k+1)\Delta} - X_{k\Delta})^4}{\Delta}.$$

We can observe that $T_{k\Delta} = \xi^4(X_{k\Delta}) \int_{\mathbf{R}} z^4\nu(dz)$+centred term+small term. We assume that $\int_{\mathbf{R}} z^4\nu(dz) = 1$, which ensures the identifiability of the model. To estimate ξ^4 non-parametrically, we consider a family of vectorial subspaces $(S_m)_{m\geq 0}$ of $L^2(A)$ and we construct a collection of estimators $\hat{\xi}^4_m$ of ξ^4 by minimising a contrast function on S_m:

$$\hat{\xi}^4_m = \arg\min_{t\in S_m} \gamma_n(t) \quad \text{where} \quad \gamma_n(t) = \frac{1}{n}\sum_{k=1}^{n} \left(t(X_{k\Delta}) - T_{k\Delta}\right)^2.$$

We prove that the risk of $\hat{\xi}^4_m$ is bounded by

$$\mathcal{R}_n(\hat{\xi}^4_m) := \mathbf{E}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\hat{\xi}^4_m(X_{k\Delta}) - \xi^4(X_{k\Delta})\right)^2\right) \leq c\|\xi^4_m - \xi^4\|^2_{L^2(A)} + \frac{\Xi D_m}{n} + C\Delta$$

where D_m is the dimension of S_m, ξ^4_m is the orthogonal projection of ξ^4 on S_m and Ξ is a constant. To construct an adaptive estimator $\hat{\xi}^4_{\hat{m}}$ of ξ^4, we minimize in m

$$\gamma_n(\hat{\xi}_m) + pen(m), \quad \text{where} \quad pen(m) = \kappa\frac{\Xi D_m}{n\Delta}$$

where κ is a universal constant. We prove that up to a multiplicative constant, the adaptive estimator $\hat{\xi}_{\hat{m}}$ satisfies an oracle inequality. Finally, we provide an upper bound $\hat{\Xi}$ for Ξ, to construct a robust estimator.

3- ESTIMATING OPTION PRICING MODELS USING A CHARACTERISTIC FUNCTION-BASED LINEAR STATE SPACE REPRESENTATION

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Options; Characteristic Function; Affine Jump-Diffusion; State Space Representation

In this paper, we develop a novel latent state filtering and parameter estimation procedure for option pricing models governed by general affine jump-diffusion (AJD) processes. Exploiting the option-spanning formula of Carr and Madan (2001) for European-style payoff functions, we imply the risk-neutral conditional characteristic function (CCF) of the underlying asset price from the option prices without imposing any parametric assumptions on the state process, that is, in a completely model-independent way. On the other hand, a large stream of literature is devoted to parametric option pricing models that often belong to the general AJD class introduced in Duffie, Pan and Singleton (2000). The defining property of the AJD class is the exponential-affine joint CCF, which is available in semi-closed form. By comparing the two option pricing representations—model-free and model-implied—we can obtain a linear relation between the logarithm of the option-implied CCF and the model-dependent CCF within the affine framework. When latent factors are present in the model specification, the linear relation enables us to exploit the rich literature on linear factor models. In particular, we extract the latent factors using a collapsed approach for the Kalman-type filter, developed by Jungbacker and Koopman (2015), and estimate model parameters based on the associated quasi-maximum likelihood method. A similar approach is often used in the affine term structure literature, where the yields themselves are affine functions of the state vector. Another advantage of our approach is that, once the model-free CCF has been obtained from the data, no further numerical option pricing methods, such as the FFT approaches or simulation-based methods, are needed, since the CCF is used to directly learn about the latent factors and model parameters. This reduces computational costs considerably relative to many existing approaches in the option pricing literature. We illustrate the developed estimation procedure in Monte Carlo simulations based on several AJD specifications, including widely used one-factor AJD option pricing models with latent stochastic volatility and jump intensity. Finally, we demonstrate our new filtering and estimation approach in an empirical application to S&P 500 index options. In particular, we filter and estimate the latent volatility from a stochastic volatility model with double-exponential jumps. We also investigate the impact of the Covid-19 propagation rate on the stock market within this model by embedding the reproduction number into the volatility and jump intensity dynamics. Our results show that while the reproduction numbers have a mild effect on total volatility, they contribute substantially to the likelihood of jumps.

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S12. Friday, July 1st, 16:00 - 17:30

1- ESTIMATING THE INTERACTION GRAPH OF STOCHASTIC NEURAL DYNAMICS BY OBSERVING ONLY PAIRS OF NEURONS (canceled)

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Neuronal networks; multivariate point processes; stochastic processes with memory of variable length; statistical model selection

We address the questions of identifying pairs of interacting neurons from the observation of their spiking activity (for a discrete version of the problem see [1]). The neuronal network is modeled by a system of interacting point processes with memory of variable length (see [2]). The influence of a neuron on another can be either excitatory or inhibitory. To identify the existence and the nature of an interaction we propose an algorithm based only on the observation of joint activity of the two neurons in successive time slots. This reduces the amount of computation and storage required to run the algorithm, thereby making the algorithm suitable for the analysis of real neuronal data sets. We obtain computable upper bounds for the probabilities of false positive and false negative detection. As a corollary we prove the consistency of the identification algorithm.

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2- EXACT AND ASYMPTOTIC ANALYSIS OF MULTIVARIATE HAWKES POPULATION PROCESSES

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Hawkes processes; mutual excitation; transform analysis; fixed-point theorem; heavy tails; transient and stationary moments

This paper considers multivariate population processes in which multivariate Hawkes processes dictate the stochastic arrivals. We establish results to determine the corresponding time-dependent joint probability distribution, allowing for general intensity decay functions, random intensity jumps and exponential sojourn times. We obtain an exact, full characterization of the time-dependent joint transform of the population process and its underlying intensity process in terms of a fixed-point representation and corresponding convergence results. We also derive the asymptotic tail behavior of the population process and its underlying intensity process in the setting of heavy-tailed intensity jumps. By exploiting the results we establish, arbitrary joint spatial-temporal moments and other distributional properties can now be readily evaluated using standard transform differentiation and inversion techniques, and we illustrate this in a few examples.

As a special case, we consider the class of exponential intensity decay functions which imply the joint processes to be Markov. Applying the Markov property directly, the time-dependent joint transform characterization is obtained in terms of a system of ODEs. We exploit this transform to derive analytic expressions for transient and stationary multivariate moments, auto- and cross-variances. Further analysis reveals a nested sequence of block matrices that yields these moments in explicit form and brings important computational advantages.

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3- STOCHASTIC MODELS OF OLFACTORY RECEPTOR NEURON RESPONSE IN A MOTH

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Olfactory system; SSA; CLE; integrate-and-fire model; Ornstein-Uhlenbeck process

Animals often rely on the sense of smell (olfaction) for detecting food or predators. Especially in some insect species, the olfaction plays a key role in finding mating partners by means of specialized chemicals (pheromones) release and detection. To understand the mechanisms underlying neuronal response to a pheromone stimulus in a species of moth, we implement a biologically relevant stochastic computational model of the ORN consisting of two main steps:

(1). Pheromone molecule binding and consequent receptor activation. We propose several extensions of the already published deterministic chemical kinetics model [1, 2]. We believe that especially at small pheromone doses (which are thought to correspond to the natural stimulation) the probabilistic nature of individual ligand-receptor interactions will affect the neuronal response significantly. For this purpose, we first investigate the Stochastic Simulation Algorithm (SSA), the Chemical Langevin Equation (CLE) and 'telegraph process' approximation to individual receptor activation .

(2). Action potential (spike) generation: The output of the previous step is integrated as a parameter in the membrane voltage equation, according to the standard integrate-and-fire model equipped with spike-frequency adaptation [1, 3, 6]

We find that SSA and CLE capture temporal fluctuations that are due to the inherent stochasticity of the biochemical reaction systems, referred to as intrinsic noise [4]. However, both methods do not consider other sources of heterogeneity in chemical processes, which are referred to as extrinsic noise, which is captured by the telegraph process approximation.

We conclude that the natural stochasticity of the pheromone binding significantly affects the dose-response relationship of the ORN neuron, allowing detection of small pheromone doses that are below the threshold of the classical deterministic model.

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